LESSON

# 13 Limits at Infinity

This lesson will serve as a preparation for the graphing in the next lesson. Here, we will work on ways to identify asymptotes from the formula of a rational function. *Rational functions* are quotients, with a clear numerator and denominator.

*Vertical asymptotes* are easy to recognize, because they occurwhere the denominator is undefined. For example, $f(x) = \frac{(3x + 2)(x - 1)}{(x + 3)(x - 4)}$ has vertical asymptotes at $x = -3$ and $x = 4$.

*Horizontal asymptotes* take a bit more work to identify. The graph will flatten out like a horizontal line if large values of $x$ all have essentially the same $y$-value.

In this graph of $y = f(x)$, for example, if $x$ is bigger than 5, then $y$ will be very close to $y = 1$ (see Figure 13.1). Thus, $y = 1$ is a horizontal asymptote. Similarly, if $x$ is a large negative number, the corresponding $y$-value will be close to zero. Thus, $y = 0$ is another horizontal asymptote. Horizontal asymptotes are related to the limits as $x$ gets really big. For $f(x)$ given in the graph:

$\lim_{x \to \infty} f(x) = 1$ and $\lim_{x \to -\infty} f(x) = 0$

## Asymptote Hint

**Notice that the graph of $y = f(x)$ crosses *both* asymptotes. Vertical asymptotes cannot be crossed because they are, by definition, not in the domain. Horizontal asymptotes *can* be crossed, as illustrated in this example. Think of "asymptote" as meaning "flattens out like a straight line" and not "a line not to be crossed."**

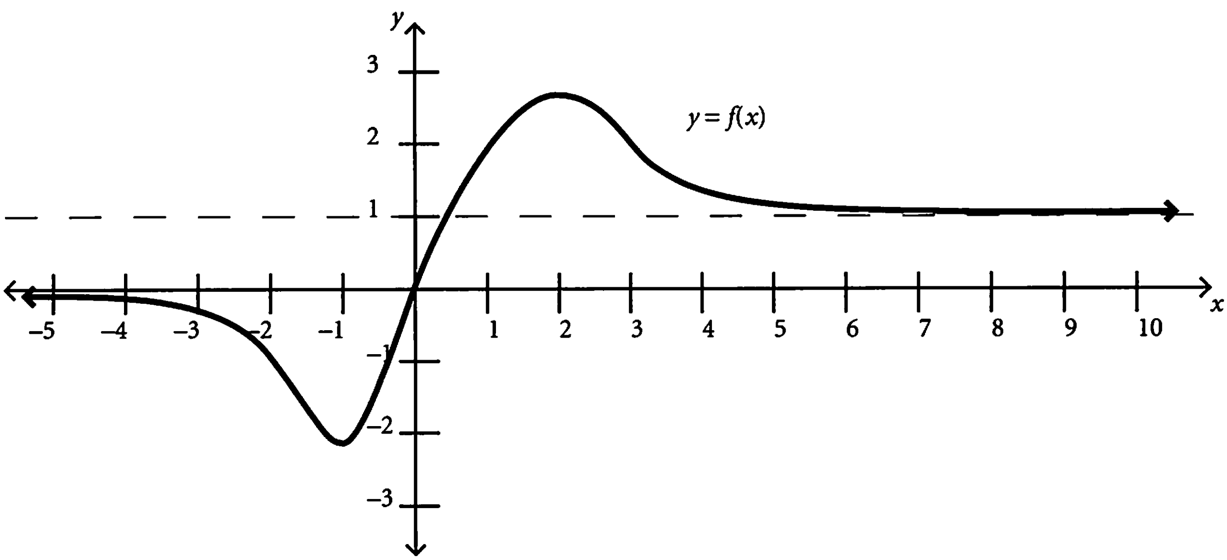



**Figure 13.1**

These *limits at infinity* (and negative infinity) identify what the ends of the graph do. For example, if $\lim_{x\to\infty} g(x) = 3$, then the graph of $y = g(x)$ will look something like that in Figure 13.2. If $\lim_{x\to-\infty} h(x) = \infty$, then the graph of $y = h(x)$ will look like that in Figure 13.3.

Notice that the infinite limits say only what happens way off to the left and to the right. Other calculations must be done to know what happens in the middle of the graph.

The general trick to evaluating an infinite limit is to focus on the most powerful part of the function. Take $\lim_{x\to\infty} 2x^3 - 100x^2 - 10x - 5{,}000$, for example. There are a lot of negative elements to this function. However, the most powerful part is the positive $2x^3$. When $x$ gets big enough, like when $x = 1{,}000{,}000$, then

$$2x^3 - 100x^2 - 10x - 5{,}000$$
$$= 2{,}000{,}000{,}000{,}000{,}000{,}000 -$$
$$100{,}000{,}000{,}000{,}000 - 10{,}000{,}000 - 5{,}000$$
$$= 1{,}999{,}899{,}999{,}989{,}995{,}000$$

This clearly rounds to 2,000,000,000,000,000,000, which is the $2x^3$. It is in this sense that

## Rules for Infinite Limits

**The rules for Infinite Limits of Rational Functions are as follows:**

- **If the numerator is more powerful, the limit goes to $\infty$ or $-\infty$.**
- **If the denominator is more powerful, the limit goes to 0.**
- **If the numerator and denominator are evenly matched, the limit is formed by the coefficients of the most powerful parts.**

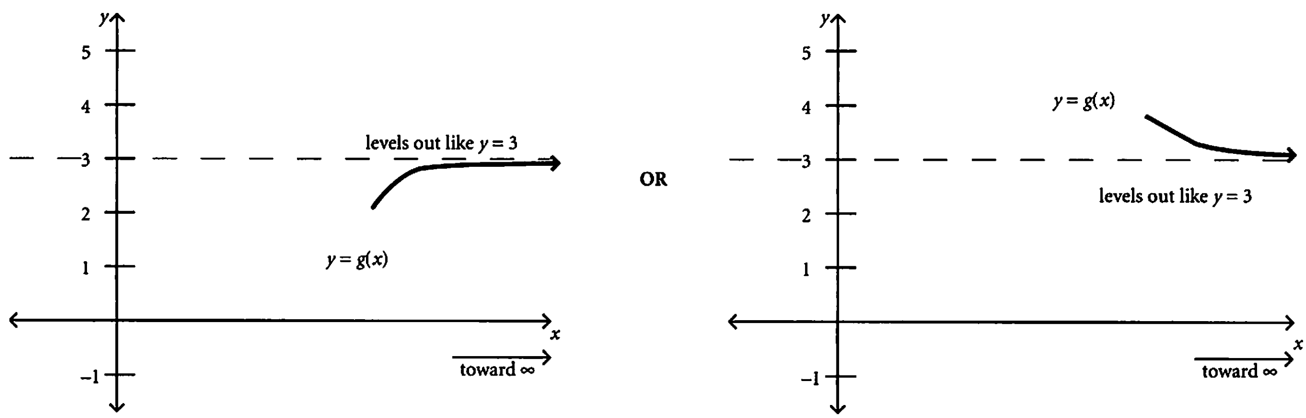



**Figure 13.2**

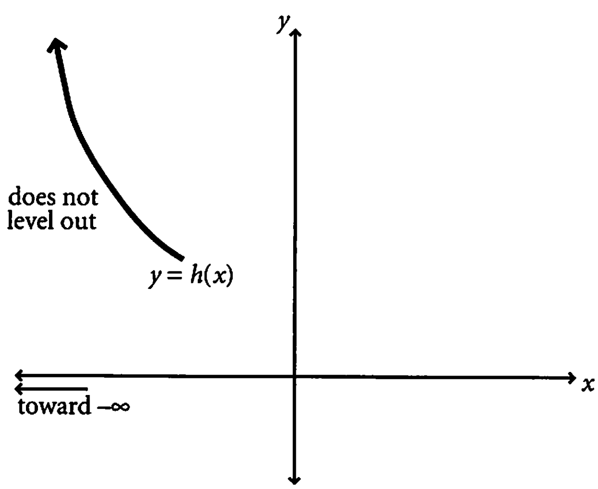



**Figure 13.3**

$2x^3$ is called the most powerful part of the function. As $x$ gets big, $2x^3$ is the only part that counts.

$$\lim_{x\to\infty} 2x^3 - 100x^2 - 10x - 5{,}000 = \lim_{x\to\infty} 2x^3 = \infty$$

As $x$ gets huge, $x^3$ is clearly even larger, and $2x^3$ is twice that. Thus, as $x$ goes to infinity, so does $2x^3$. Basically, the higher the exponent of $x$, the more powerful it is. With that in mind, the rules for infinite limits of rational functions are fairly simple:

- If the numerator is more powerful, the limit goes to $\infty$ or $-\infty$.
- If the denominator is more powerful, the limit goes to 0.
- If the numerator and denominator are evenly matched, the limit is formed by the coefficients of the most powerful parts.

## Going to Infinity

The whole concept of "going to infinity" might be a bit confusing. This really means "going toward infinity," because infinity is not something that a real number can reach. So don't wander off pondering the one number that is bigger than all the rest (unless you enjoy that). Just know that "going to infinity" means using really big numbers, and that "going to negative infinity" means using really big negative numbers.

### Example

Evaluate $\lim_{x\to\infty}\frac{1-x^2}{x^3+3x+2}$.

### Solution

The most powerful part of the numerator is $-x^2$, and in the denominator is $x^3$. Thus:

$$\lim_{x\to\infty}\frac{1-x^2}{x^3+3x+2} = \lim_{x\to\infty}\frac{-x^2}{x^3} = \lim_{x\to\infty} -\frac{1}{x} = 0$$

This goes to zero because the numerator is clearly outclassed by the more powerful denominator. Also, as $x$ gets really big, $\frac{1}{x}$ gets really close to zero. For example, when $x = 1{,}000$, then $\frac{1}{x} = \frac{1}{1{,}000} = 0.001$.

### Example

Evaluate $\lim_{x\to-\infty}\frac{3x^2+2x-5}{1-8x^2}$.

### Solution

Here, the numerator and denominator are evenly matched, with each having $x^2$ as its highest power of $x$.

$$\lim_{x\to-\infty}\frac{3x^2+2x-5}{1-8x^2} = \lim_{x\to-\infty}\frac{3x^2}{-8x^2}$$
$$= \lim_{x\to-\infty} -\frac{3}{8} = -\frac{3}{8}$$

The limit is formed by the coefficients of the most powerful parts: 3 in the numerator and $-8$ in the denominator.

### Example

Evaluate $\lim_{x\to\infty}\frac{5x^{10}-4x^5+7}{1-x^2}$.

### Solution

Here,

$$\lim_{x\to\infty}\frac{5x^{10}-4x^5+7}{1-x^2} = \lim_{x\to\infty}\frac{5x^{10}}{-x^2}$$
$$= \lim_{x\to\infty} -5x^8 = -\infty$$

As $x$ goes to infinity, $x^8$ also gets really large, but the negative in the $-5$ reverses this and makes $-5x^8$ approach negative infinity.

## ▶ Practice

Evaluate the following infinite limits.

**1.** $\lim_{x\to\infty}\frac{5x^3+10x^2-2}{8x^4+1}$

**2.** $\lim_{x\to-\infty}\frac{4x^3+10x^2+3x}{5x^3+8x-1}$

**3.** $\lim_{x\to\infty}\frac{5x+2}{2x-1}$

**4.** $\lim\limits_{x\to\infty} \dfrac{10x^3 - 3x - 100}{2x + 5}$

**5.** $\lim\limits_{t\to-\infty} \dfrac{t + 1}{t^3 + 3t - 4}$

**6.** $\lim\limits_{t\to\infty} \dfrac{8t^4 - 3t^3 + 11}{1 - 9t^4}$

**7.** $\lim\limits_{x\to\infty} \dfrac{5x^3 + x - 9}{1 - x^2}$

**8.** $\lim\limits_{x\to-\infty} \dfrac{x^4 + 3x^2 - 8x + 4}{x^2 + 2x + 1}$

**9.** $\lim\limits_{x\to-\infty} \dfrac{x^2 - 1}{x^2 + 1}$

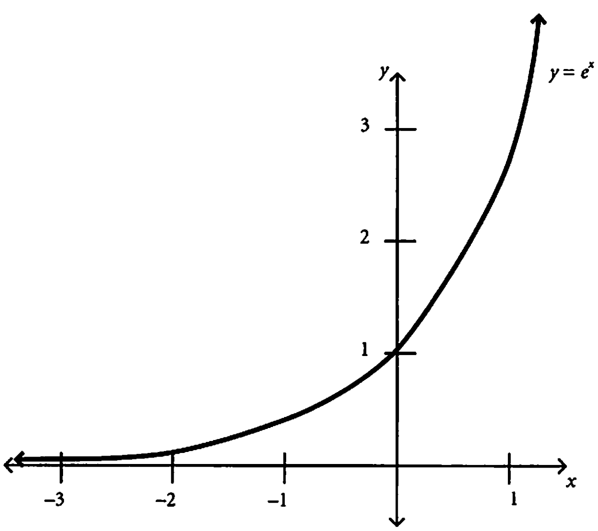
**10.** $\lim\limits_{x\to\infty} \dfrac{t^{10} + 4t^2 - 11}{2t^{15} - 7t}$

The infinite limits of $e^x$ and $\ln(x)$ can be seen from their graphs in Figure 13.4.

$$\lim_{x\to\infty} e^x = \infty \qquad \lim_{x\to-\infty} e^x = 0 \qquad \lim_{x\to\infty} \ln(x) = \infty$$

In general, as $x$ goes to infinity, $e^x$ is more powerful than $x$ raised to any number. The natural logarithm, however, goes to infinity slower than just about anything else. It may look as though $y = \ln(x)$ is beginning to level out into a horizontal asymptote, but actually, it will eventually surpass any height as it slowly goes up to infinity.

In more complicated situations, we use L'Hôpital's rule. This states that if the numerator and denominator both go to infinity (positive or negative), then the limit remains the same after taking the derivative of the top and the bottom.



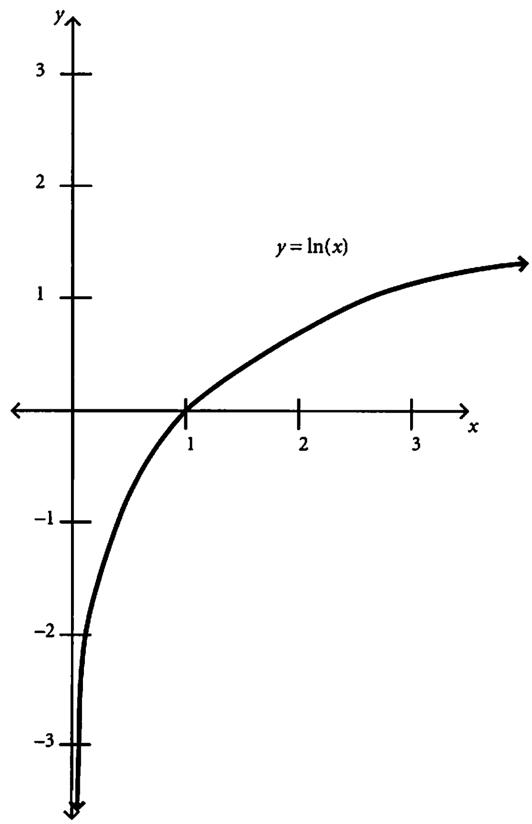




Figure 13.4

## L'Hôpital's Rule

If the numerator and denominator both go to infinity (positive or negative), the limit remains the same after taking the derivative of the top and bottom, OR:

$$\lim_{x\to\pm\infty}\frac{f(x)}{g(x)} = \lim_{x\to\pm\infty}\frac{f'(x)}{g'(x)} \text{ if } \lim_{x\to\pm\infty} f(x) = \pm\infty \text{ and } \lim_{x\to\pm\infty} g(x) = \pm\infty$$

### Example

Evaluate $\lim_{x\to\infty}\frac{\ln(x)}{1-x}$.

### Solution

Since $\lim_{x\to\infty}\ln(x) = \infty$ and $\lim_{x\to\infty} 1 - x = -\infty$, we can use L'Hôpital's Rule.

$$\lim_{x\to\infty}\frac{\ln(x)}{1-x} \overset{H}{=} \lim_{x\to\infty}\frac{\frac{d}{dx}(\ln(x))}{\frac{d}{dx}(1-x)}$$

$$= \lim_{x\to\infty}\frac{\frac{1}{x}}{-1} = \lim_{x\to\infty} -\frac{1}{x} = 0$$

**Note:** The little $H$ over the equals sign indicates that L'Hôpital's Rule as been used at that point. Examples like this demonstrate how $\ln(x)$ goes to infinity even slower than $x$ does.

### Example

Evaluate $\lim_{x\to\infty}\frac{e^x}{x^3+2x^2+5x+2}$.

### Solution

Here, $\lim_{x\to\infty} e^x = \infty$ and $\lim_{x\to\infty} x^3 + 2x^2 + 5x + 2 = \infty$, so we use L'Hôpital's Rule.

$$\lim_{x\to\infty}\frac{e^x}{x^3+2x^2+5x+2}$$

$$\overset{H}{=} \lim_{x\to\infty}\frac{\frac{d}{dx}(e^x)}{\frac{d}{dx}(x^3+2x^2+5x+2)}$$

$$= \lim_{x\to\infty}\frac{e^x}{3x^2+4x+5}$$

Notice that we don't use the Quotient Rule, because we take the derivative of the numerator and denominator separately. Here, we need to use L'Hôpital's Rule several more times:

$$\lim_{x\to\infty}\frac{e^x}{3x^2+4x+5} \overset{H}{=} \lim_{x\to\infty}\frac{e^x}{6x+4}$$

$$\overset{H}{=} \lim_{x\to\infty}\frac{e^x}{6} = \infty$$

This example shows how $e^x$ is more powerful than $x^3$. If the denominator had an $x^{100}$, we'd have to use L'Hôpital's Rule 100 times, but in the end, the $e^x$ would take everything to infinity.

### Example

Evaluate $\lim\limits_{x \to -\infty} \frac{e^x}{x^5 + 7x - 1}$.

### Solution

This is a trick question! The limit $\lim\limits_{x \to -\infty} e^x = 0$ is not infinite, so we *can't* use L'Hôpital's Rule. The function $e^x$ is only powerful when $x$ goes to positive infinity. Instead, we use the old "plug in" method (or common sense).

$$\lim_{x \to -\infty} \frac{e^x}{x^5 + 7x - 1} = \frac{0}{\text{something not zero}} = 0$$

### Example

Evaluate $\lim\limits_{x \to \infty} \frac{\sin(x)}{x^2}$.

### Solution

This has the same problem as the previous example. No matter what $x$ may be, $\sin(x)$ will always be between $-1$ and 1. Thus, $-1 \leq \sin(x) \leq 1$ and so

$$\frac{-1}{x^2} \leq \frac{\sin(x)}{x^2} \leq \frac{1}{x^2}$$

Because $\lim\limits_{x \to \infty} \frac{1}{x} = 0$ and $\lim\limits_{x \to \infty} \frac{-1}{x} = 0$, the function $\frac{\sin(x)}{x^2}$ is squeezed between them to zero as well: $\lim\limits_{x \to \infty} \frac{\sin(x)}{x^2} = 0$. This is called the *Squeeze Theorem* or the *Sandwich Theorem* because of the way $\frac{\sin(x)}{x^2}$ is squished between something above it going to zero and something below it going to zero.

## ▶ Practice

Evaluate the following limits.

**11.** $\lim\limits_{x \to \infty} \frac{\ln(x^3)}{\ln(x) + 5}$

**12.** $\lim\limits_{x \to \infty} \frac{x + 5}{\sqrt{x} - 1}$

**13.** $\lim\limits_{x \to -\infty} \frac{x^2 + 5x - 10}{4x + 2}$

**14.** $\lim\limits_{x \to \infty} \frac{3x^2 + 2}{x - \ln(x)}$

**15.** $\lim\limits_{x \to -\infty} \frac{4x^3 - 10x + 7}{15x^3 + 4x^2 - 2x + 1}$

**16.** $\lim\limits_{x \to \infty} \frac{4x + 6}{3x^2 - 2x + 5}$

**17.** $\lim\limits_{x \to \infty} \frac{3x + 7}{\ln(x)}$

**18.** $\lim\limits_{x \to \infty} \frac{\cos(x)}{x}$

**19.** $\lim\limits_{x \to \infty} \frac{4x^3 + 5x^2 + 2}{e^x - 7x^3}$

**20.** $\lim\limits_{x \to -\infty} \frac{4x^3 + 5x^2 + 2}{e^x - 7x^3}$

## ▶ Sign Diagrams

In order to calculate the limits at vertical asymptotes, it is necessary to know where the function is positive and negative. The key to everything is this: A continuous function cannot switch between positive and negative without being zero or undefined. Functions are

generally zero when the numerator is zero and undefined where the denominator is zero. Mark these points on a number line. Between these points, the function must be entirely positive or negative. This can be found by testing any point in each interval.

For example, consider $f(x) = \frac{x-4}{(x+2)(1-x)}$. This function is zero at $x = 4$ and undefined at both $x = -2$ and $x = 1$. We mark these on a number line (see Figure 13.5).

In between $x = -2$ and $x = 1$, the function is either always positive or always negative. To find out which it is, we test a point between $-2$ and 1, such as 0. Because $f(0) = \frac{-4}{2(1)} = -2$ is negative, the function is always negative between $-2$ and 1. Similarly, we check a point between 1 and 4, such as $f(2) = \frac{-2}{4(-1)} = \frac{1}{2}$, a point after 4, such as $f(5) = \frac{1}{7(-4)} = -\frac{1}{28}$, and a point before $-2$, such as $f(-3) = \frac{-7}{-1(4)} = \frac{7}{4}$. These calculations can be made very roughly, because it matters only if the function is positive or negative at the selected point. In any case, the *sign diagram* for this function is shown in Figure 13.6.

This makes calculating the limits at the vertical asymptotes very easy. Not only does $f(x) = \frac{x-4}{(x+2)(1-x)}$ have vertical asymptotes at $x = -2$ and $x = 1$, but the limits are:

$$\lim_{x \to -2^-} \frac{x-4}{(x+2)(1-x)} = \infty$$

$$\lim_{x \to -2^+} \frac{x-4}{(x+2)(1-x)} = -\infty$$

$$\lim_{x \to 1^-} \frac{x-4}{(x+2)(1-x)} = -\infty$$

$$\lim_{x \to 1^+} \frac{x-4}{(x+2)(1-x)} = \infty$$

At the same time, we can calculate the limits at infinity:

$$\lim_{x \to \infty} \frac{x-4}{(x+2)(1-x)} = \lim_{x \to \infty} \frac{x-4}{-x^2 - x + 2} = 0$$

$$\lim_{x \to -\infty} \frac{x-4}{(x+2)(1-x)} = 0$$

Thus, $f(x)$ has a horizontal asymptote of $y = 0$. With all of this, we begin to get a picture of $f(x) = \frac{x-4}{(x+2)(1-x)}$, which can be seen in Figure 13.7.

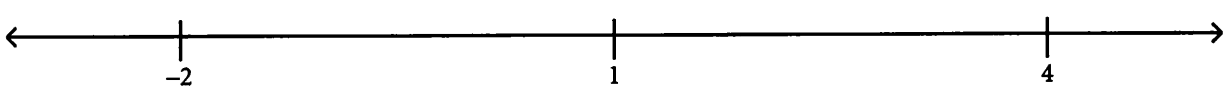



**Figure 13.5**

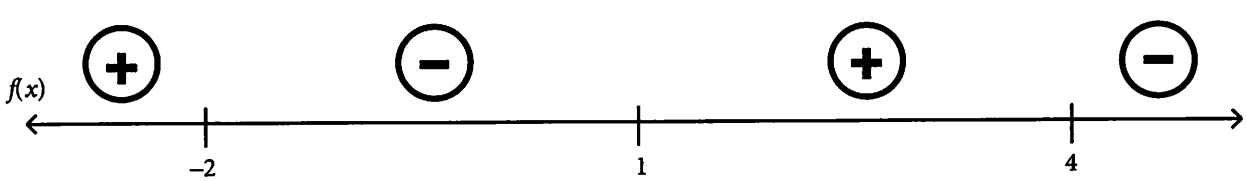



**Figure 13.6**

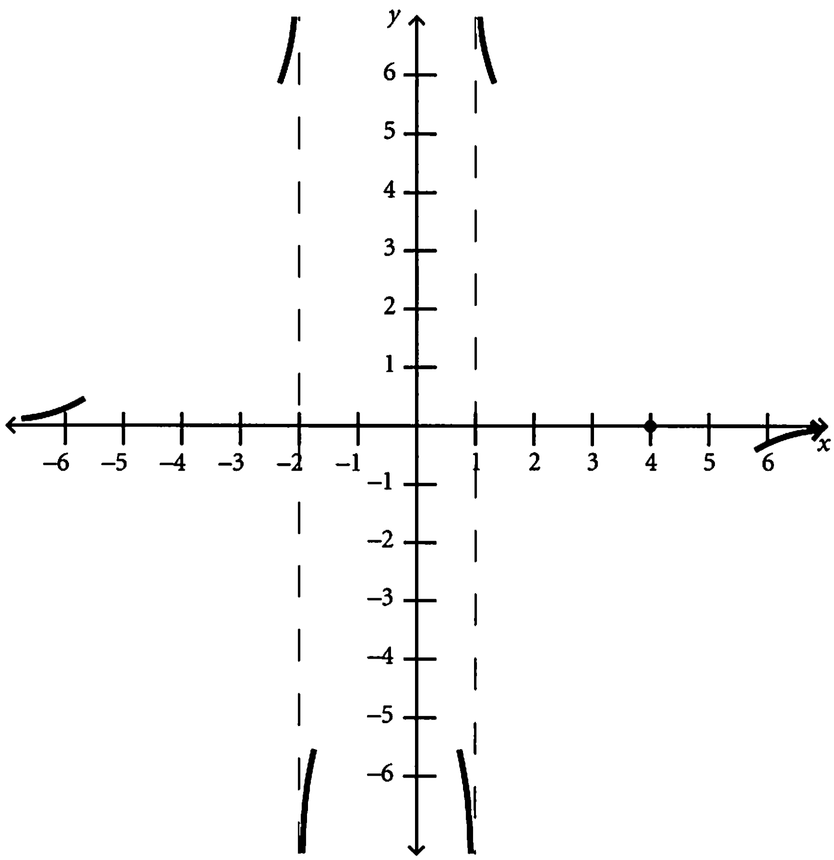



**Figure 13.7**

Notice that the horizontal asymptote $y = 0$ is approached from above as $x \to -\infty$, because $f(x)$ is always positive when $x < -2$. At the other end, the asymptote is approached from below as $x \to \infty$ because the function is negative when $x > 4$.

We shall deal with graphing more thoroughly in the next lesson.

## ▶ Practice

Name all the asymptotes, vertical and horizontal, of the following functions. Also, make a sign diagram for each.

**21.** $f(x) = \dfrac{x+2}{x-4}$

**22.** $g(x) = \dfrac{x-3}{x^2-4}$

**23.** $h(x) = \dfrac{x^2 - 1}{(x + 3)^2}$

**24.** $k(x) = \dfrac{2x + 1}{x^2 - 4x + 3}$

Evaluate the following limits.

**25.** $\lim\limits_{x\to 4^+} \dfrac{x + 2}{x - 4}$

**26.** $\lim\limits_{x\to 2^-} \dfrac{x - 3}{x^2 - 4}$

**27.** $\lim\limits_{x\to -3^-} \dfrac{x^2 - 1}{(x + 3)^2}$

**28.** $\lim\limits_{x\to 3^-} \dfrac{x + 1}{x^2 - 4x + 3}$